

Polynomials with small norm on compact Riemannian homogeneous manifolds

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Abstract

Let H_k , $k \geq 0$ be the sequence of eigenspaces corresponding to the eigenvalues $0 \leq \theta_0 \leq \theta_1 \leq \dots \leq \theta_n \leq \dots$ of the Laplace-Beltrami operator Δ on a compact Riemannian homogeneous manifold \mathbb{M}^d with the normalized invariant measure ν and $\mathcal{T}_n = \bigoplus_{k=0}^n H_k$. We consider the problem of existence of polynomials $t_n \in \mathcal{T}_n$ with small norm. Namely, we show that for any $\epsilon \in (0, 1)$ and any subspace $L_m \subset \mathcal{T}_n$, $\dim L_m \geq \epsilon n$, there exists such $t_n \in L_m$ that $\|t_n\|_{L_p(\mathbb{M}^d, \nu)} \leq C_{p,q} \|t_n\|_{L_q(\mathbb{M}^d, \nu)}$, where $C_{p,q}$ depends just on p and q , $1 < q < p < \infty$. In the case $p = \infty$ or $q = 1$ an extra logarithmic factor appears. This range of problems has been extensively studied by many authors in the case $\mathbb{M}^d = \mathbb{T}^1$, the unit circle (or compact Abelian group \mathcal{G}), i.e., when the characters of \mathcal{G} are bounded by 1. In general, on compact Riemannian homogeneous manifolds, the eigenfunctions of the Laplace-Beltrami operator are not uniformly bounded that creates difficulties of a fundamental nature in applications of known methods and results. The method, we develop, is based on a geometric inequality between norms induced by two convex bodies in \mathbb{R}^n .

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1 Introduction

The range of problems we consider in this article has been traditionally studied in the context of random Fourier series and has been initiated in the classical works of Paley and Zygmund. In many situations it is difficult or impossible to give explicitly an example of a certain object having a required property and frequently one gets by with Lebesgue measure [11], [12]. Problems regarding flat polynomials with coefficients ± 1 whose uniform norm is close to their $L_2(\mathbb{T}^1)$

norm has attracted a lot of attention [4], [2]. It was shown in [22] that for any $N \in \mathbb{N}$ there is a sequence $\epsilon_n = \pm 1$, $1 \leq n \leq N$ such that

$$\left| \sum_{k=1}^N \epsilon_n e^{in\theta} \right| < 5N^{1/2}.$$

This topic has been developed in [13]. It was shown that for all $|z| = 1$ there is a sequence of polynomials $P(z) = \sum_{m=1}^N a_{m,n} z^m$, $|a_{m,n}| = 1$ such that

$$(1 - \epsilon_N)N^{1/2} \leq |P_N(z)| \leq (1 + \epsilon_N)N^{1/2},$$

where $\epsilon_N \leq CN^{-1/17}(\log N)^{1/2}$ as $N \rightarrow \infty$. The expected $L_p(\mathbb{T}^1)$ norm of random trigonometric polynomials $q_N(\theta) = \sum_{k=0}^N X_k e^{ik\theta}$, where X_k , $k \geq 0$ are independent and identically distributed random variables with mean 0 and variance 1 was studied in [3]. In particular, it was shown that

$$\frac{\mathbb{E}(\|q_N\|_p^p)}{N^{p/2}} \rightarrow \Gamma\left(1 + \frac{p}{2}\right), \quad N \rightarrow \infty.$$

The problem of existence of trigonometric polynomials with special properties of degree $\leq (M+1)(1+\epsilon)$, $\epsilon > 0$, in any subspace of $L_2(\mathbb{T}^1)$ of codimension M was considered in [14]. It was shown that for any $\epsilon > 0$ there is such a polynomial whose uniform norm is 1, and such that the sum of the absolute values of the coefficients is at least $c_\epsilon M^{1/2}$.

In this article we consider the problem of existence of polynomials on a compact homogeneous Riemannian manifold \mathbb{M}^d whose $L_p(\mathbb{M}^d)$ norm is close to their $L_2(\mathbb{M}^d)$ norm for any $1 \leq p \leq \infty$ (see Theorem 3).

The method's possibilities are not confined to the theorem proved in the Section 4 but can be used in studying more general problems. The results we derive are apparently new even in the one dimensional case.

2 Elements of Harmonic Analysis on Compact Riemannian Manifolds

First we give a general definition of function spaces that we consider and then present various important examples.

Definition 1. *Given a measure space (Ω, ν) . Let $\Xi = \{\xi_k\}_{k \in \mathbb{N}}$ be a set of orthonormal functions in $L_2(\Omega, \nu)$. Suppose that there exists a sequence $\{k_j\}_{j \in \mathbb{N}}$, $k_1 = 1$, such that for any $j \in \mathbb{N}$ and some $C > 0$*

$$\sum_{k=k_j}^{k_{j+1}-1} |\xi_k(x)|^2 \leq C d_j$$

a.e. on Ω , where $d_j = k_{j+1} - k_j$. Then we say that $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$.

Let $L_p = L_p(\Omega, \nu)$ be the usual set of p -integrable functions on Ω . Suppose that $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$. Since all the functions ξ_k are a.e. bounded on Ω , then for an arbitrary function $\phi \in L_p$, $1 \leq p \leq \infty$ we can construct the sequence $\{c_k(\phi)\}_{k \in \mathbb{N}}$, where $c_k(\phi) = \int_{\Omega} \phi \overline{\xi_k} d\nu$ and consider the formal series

$$\phi \sim \sum_{l=1}^{\infty} \sum_{k_l}^{k_{l+1}-1} c_k(\phi) \xi_k.$$

The family \mathcal{K} is sufficiently large. We consider compact, connected, orientable, d -dimensional C^∞ Riemannian manifold \mathbb{M}^d , with C^∞ metric. Let g its metric tensor, ν its normalized volume element and Δ its Laplace-Beltrami operator. In local coordinates x_l , $1 \leq l \leq d$,

$$\Delta = -(\overline{g})^{-1/2} \sum_k \frac{\partial}{\partial x_k} \left(\sum_j g^{jk}(\overline{g})^{1/2} \frac{\partial}{\partial x_j} \right).$$

Here, $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$, $\overline{g} = |\det(g_{ij})|$ and $(g^{ij}) = (g_{ij})^{-1}$. It is well-known that Δ is an elliptic, self adjoint, invariant under isometries, second-order operator. The eigenvalues θ_k , $k \geq 0$ of Δ are discrete, nonnegative and form an increasing sequence $0 \leq \theta_0 \leq \theta_1 \leq \dots \leq \theta_n \leq \dots$ with $+\infty$ the only accumulation point. Corresponding eigenspaces H_k , $k \geq 0$ are finite dimensional, $d_k = \dim H_k < \infty$, $k \geq 0$, orthogonal and $L_2(M^d, \nu) = \oplus_{k=0}^{\infty} H_k$. Let $\{Y_m^k\}_{m=1}^{d_k}$ be an orthonormal basis of H_k , $H_k = \text{lin}\{Y_m^k\}_{m=1}^{d_k}$.

Recall that a Riemannian manifold \mathbb{M}^d is called homogeneous is its group of isometries \mathcal{G} acts transitively on it. Let H_j , $j \geq 0$ be any eigenspace of Δ , $d_j = \dim H_j$, f_1, \dots, f_{d_j} any orthonormal basis of H_j , then

$$\sum_{s=1}^{d_j} |f_s(x)|^2 = d_j,$$

for any $x \in \mathbb{M}^d$ (see, e.g., [8]). Hence, any compact, connected, orientable, d -dimensional C^∞ , homogeneous Riemannian manifold \mathbb{M}^d , with C^∞ metric has the property \mathcal{K} . Here we give several important examples of such manifolds:

1. A Grassmannian (Grassmann manifold), $\mathbb{G}_{m,n}(\mathbb{R})$ is the space of all m -dimensional subspaces of \mathbb{R}^n . Grassmann manifold also appear as coset space $\mathbb{G}_{m,n}(\mathbb{R}) = \text{O}(n)/\text{O}(n-m) \times \text{O}(m)$;
2. A complex Grassmannian manifold $\mathbb{G}_{m,n}(\mathbb{C})$ is the space of all m -dimensional complex subspaces in \mathbb{C}^n ;
3. An n -torus, \mathbb{T}^d is defined as a product of n circles: $\mathbb{T}^d = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$. The n -torus can be described as a quotient of \mathbb{R}^n under shifts in any coordinate. That is, the n -torus is \mathbb{R}^n modulo the action of the integer lattice \mathbb{Z}^n (with the action being taken as vector addition);
4. The Stiefel manifold, denoted $\mathbb{V}_k(\mathbb{R}^d)$ or $\mathbb{V}_{k,d}$, is the set of all orthonormal k -frames in \mathbb{R}^d . That is, it is the set of ordered k -tuples of orthonormal vectors

in \mathbb{R}^d . When $k = 1$, the manifold $\mathbb{V}_{1,d}$ is just the set of unit vectors in \mathbb{R}^d ; that is, $\mathbb{V}_{1,d}$ is diffeomorphic to the $d - 1$ sphere, \mathbb{S}^{d-1} . At the other extreme, when $k = d$, the Stiefel manifold $\mathbb{V}_{d,d}$ is the set of all ordered orthonormal bases for \mathbb{R}^d . $\mathbb{V}_{d,d}$ is a principal homogeneous space for $O(d)$ and therefore diffeomorphic to it. In general, the orthogonal group $O(d)$ acts transitively on $\mathbb{V}_{k,d}$ with stabilizer subgroup isomorphic to $O(d - k)$. Therefore $\mathbb{V}_{k,d}$ can be viewed as the homogeneous space $\mathbb{V}_{k,d} = O(d)/O(d - k)$.

5. The unit complex sphere in \mathbb{C}^d is defined as $\mathbb{S}_{\mathbb{C}}^d = \{z \in \mathbb{C}^d \mid \langle z, z \rangle = 1\}$, where $\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_d \bar{w}_d$, $z, w \in \mathbb{C}^d$.

6. A Riemannian manifold is two-point homogeneous if for any set of four points x_1, y_1, x_2, y_2 with $d(x_1, y_1) = d(x_2, y_2)$, d being the Riemannian metric on \mathbb{M}^d , there exists $\phi \in \mathcal{G}$ such that $\phi(x_1) = x_2$ and $\phi(y_1) = y_2$. A complete classification of the two-point homogeneous spaces was given in [23]. For information on this classification see, e.g., [6, 7, 9, 10]. They are: the spheres \mathbb{S}^d , $d = 1, 2, 3, \dots$; the real projective spaces $\mathbb{P}^d(\mathbb{R})$, $d = 2, 3, 4, \dots$; the complex projective spaces $\mathbb{P}^d(\mathbb{C})$, $l = d/2$, $d = 4, 6, 8, \dots$; the quaternionic projective spaces $\mathbb{P}^d(\mathbb{H})$, $d = 8, 12, \dots$; the Cayley elliptic plane $P^{16}(\text{Cay})$. The superscripts here denote the dimension over the reals of the underlying manifolds \mathbb{M}^d .

3 A geometric inequality

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$ and $\langle \alpha, \beta \rangle = \sum_{k=1}^n \alpha_k \beta_k$. Let $\|\alpha\|_{(2)} = \langle \alpha, \alpha \rangle^{1/2}$ be the Euclidean norm on \mathbb{R}^n , $\mathbb{S}^{n-1} = \{\alpha \in \mathbb{R}^n : \|\alpha\|_{(2)} = 1\}$ be the unit sphere in \mathbb{R}^n , $B_{(2)}^n = \{\alpha \in \mathbb{R}^n : \|\alpha\|_{(2)} \leq 1\}$ be the unit ball in \mathbb{R}^n and Vol_n be the standard n -dimensional volume of subsets in \mathbb{R}^n . Let us fix a norm $\|\cdot\|$ on \mathbb{R}^n and denote by E the Banach space $E = (\mathbb{R}^n, \|\cdot\|)$ with the ball $B_E = V$. The Lévy mean M_V is defined by

$$M = M(\mathbb{R}^n, \|\cdot\|) = \int_{\mathbb{S}^{n-1}} \|\alpha\| d\mu(\alpha).$$

For a convex centrally symmetric body $V \subset \mathbb{R}^n$ we define the polar body V° of V as

$$V^\circ = \left\{ \alpha \in \mathbb{R}^n : \sup_{\beta \in V} |\langle \alpha, \beta \rangle| \leq 1 \right\}.$$

The dual space $E^\circ = (\mathbb{R}^n, \|\cdot\|_o)$ is endowed with the norm

$$\|\alpha\|_{V^\circ} = \|\alpha\|_o = \sup_{\beta \in B_E} |\langle \alpha, \beta \rangle|$$

and $B_{E^\circ} = V^\circ$.

Theorem 1. *Let V and W be any convex symmetric bodies in \mathbb{R}^n , $V \subset B_{(2)}^n$, then for any $n \in \mathbb{N}$ and $\epsilon > 0$ there is such $0 < \mu_\epsilon < 1$ that in any subspace $L_m \subset \mathbb{R}^n$, $m = \dim L_m \geq \mu_\epsilon n$ there is such $\alpha^* \in L_m$ that*

$$\frac{\|\alpha^*\|_V}{\|\alpha^*\|_W} \leq C_\epsilon (M_V)^{1+\epsilon} M_{W^\circ}.$$

where C_ϵ depends just on ϵ .

Proof From the Urysohn inequality (see, e.g., [21] p. 6-7) it follows that

$$\begin{aligned} \left(\frac{\text{Vol}_n(V)}{\text{Vol}_n(B_{(2)}^n)} \right)^{1/n} &\leq C_1 \int_{\mathbb{S}^{n-1}} \|\alpha\|_{V^\circ} d\mu(\alpha) = C_1 M_{V^\circ} \\ &\leq C_1 \left(\int_{\mathbb{S}^{n-1}} \|\alpha\|_{V^\circ}^2 d\mu \right)^{1/2} = C_1 M_{V^\circ} \end{aligned}$$

or

$$\text{Vol}_n(V^\circ) \leq C_1^n (M_V)^n \text{Vol}_n(B_{(2)}^n).$$

Comparing the last estimate with the Bourgain-Milman inequality [1] p. 320,

$$\left(\frac{\text{Vol}_n(V) \cdot \text{Vol}_n(V^\circ)}{\left(\text{Vol}_n(B_{(2)}^n) \right)^2} \right)^{1/n} \geq C_2,$$

we get

$$\text{Vol}_n(V) \geq \left(\frac{C_2}{C_1 M_V} \right)^n \text{Vol}_n(B_{(2)}^n). \quad (1)$$

Let W be any convex symmetric body in \mathbb{R}^n . Using isoperimetric inequality on \mathbb{S}^{n-1} it is possible to show [1] that for every $0 < \lambda < 1$ there exists a subspace $L_{m_1} \subset \mathbb{R}^n$ with $m_1 = \dim L_{m_1} \geq \lambda n$ such that for any $\alpha \in L_{m_1}$ we have

$$\|\alpha\|_{(2)} \leq C_3 \frac{M_{W^\circ}}{(1-\lambda)} \|\alpha\|_W. \quad (2)$$

Let $L_{m_2} \subset \mathbb{R}^n$ be any m_2 -dimensional subspace. Assume that $m_1 + m_2 > n$, so that $L_{m_1} \cap L_{m_2} \neq \emptyset$ and

$$m_3 := \dim(L_{m_1} \cap L_{m_2}) \geq m_1 + m_2 - n.$$

Let $(L_{m_1} \cap L_{m_2})^\perp$ be the orthogonal complement of $L_{m_1} \cap L_{m_2}$ and $P_{(L_{m_1} \cap L_{m_2})^\perp}(V)$ be the orthogonal projection of V onto $(L_{m_1} \cap L_{m_2})^\perp$. Assume that $V \subset B_{(2)}^n$, then

$$P_{(L_{m_1} \cap L_{m_2})^\perp}(V) \subset P_{(L_{m_1} \cap L_{m_2})^\perp}(B_{(2)}^{m_3})$$

and

$$\text{Vol}_{m_3}(P_{(L_{m_1} \cap L_{m_2})^\perp}(V)) \leq \text{Vol}_{m_3}(P_{(L_{m_1} \cap L_{m_2})^\perp}(B_{(2)}^{m_3})).$$

Hence,

$$\text{Vol}_n(V) = \int_V dx = \int_{P_{(L_{m_1} \cap L_{m_2})^\perp}(V)} \text{Vol}_{m_3}(V \cap (y + L_{m_1} \cap L_{m_2})) dy.$$

Thus, for any $y \in P_{(L_{m_1} \cap L_{m_2})^\perp}(V)$ by the Brunn-Minkowski theorem

$$\text{Vol}_{m_3}(V \cap (y + L_{m_1} \cap L_{m_2})) \leq \text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2})).$$

and, therefore,

$$\begin{aligned}\text{Vol}_n(V) &\leq \text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2})) \cdot \text{Vol}_{n-m_3}(P_{(L_{m_1} \cap L_{m_2})^\perp}(V)) \\ &\leq \text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2})) \cdot \text{Vol}_{n-m_3}(B_{(2)}^{n-m_3})\end{aligned}\quad (3)$$

Comparing (1) and (3) we find that for any convex symmetric body $V \subset B_{(2)}^n$ and any m_3 -dimensional subspace $L_{m_1} \cap L_{m_2} \subset \mathbb{R}^n$,

$$\text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2})) \geq \left(\frac{C_2}{C_1 M_V} \right)^n \frac{\text{Vol}_n(B_{(2)}^n)}{\text{Vol}_{n-m_3}(B_{(2)}^{n-m_3})}.\quad (4)$$

Applying the Santalo inequality (see, e.g. [5])

$$\frac{\text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2})) \cdot \text{Vol}_{m_3}((V \cap (L_{m_1} \cap L_{m_2}))^o)}{(\text{Vol}_{m_3}(B_{(2)}^{m_3}))^2} \leq 1$$

we obtain

$$\text{Vol}_{m_3}((V \cap (L_{m_1} \cap L_{m_2}))^o) \leq \frac{(\text{Vol}_{m_3}(B_{(2)}^{m_3}))^2}{\text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2}))}.$$

Combining this result with the Bieberbach inequality (see, e.g., [5])

$$\begin{aligned}2^{m_3} \text{Vol}_{m_3}(B_{(2)}^{m_3}) (\text{diam}(V \cap (L_{m_1} \cap L_{m_2})))^{-m_3} \\ \leq \text{Vol}_{m_3}((V \cap (L_{m_1} \cap L_{m_2}))^o),\end{aligned}$$

we get the lower bound for the diameter of the set $V \cap (L_{m_1} \cap L_{m_2})$,

$$\begin{aligned}\text{diam}(V \cap (L_{m_1} \cap L_{m_2})) &\geq 2 \left(\frac{\text{Vol}_{m_3}(B_{(2)}^{m_3})}{\text{Vol}_{m_3}((V \cap (L_{m_1} \cap L_{m_2}))^o)} \right)^{1/m_3} \\ &\geq 2 \left(\frac{\text{Vol}_{m_3}(V \cap (L_{m_1} \cap L_{m_2}))}{\text{Vol}_n(B_{(2)}^n)} \right)^{1/m_3}.\end{aligned}\quad (5)$$

Comparing (4) and (5) we find

$$\text{diam}(V \cap (L_{m_1} \cap L_{m_2})) \geq \left(\frac{2C_2}{C_1 M_V} \right)^{n/m_3} \omega_{n,m_3},\quad (6)$$

where

$$\omega_{n,m_3} := \left(\frac{\text{Vol}_n(B_{(2)}^n)}{\text{Vol}_{n-m_3}(B_{(2)}^{n-m_3}) \cdot \text{Vol}_{m_3}(B_{(2)}^{m_3})} \right)^{1/m_3}.$$

Recall that

$$\text{Vol}_n(B_{(2)}^n) = 2\pi^{n/2} \Gamma\left(\frac{n}{2} + 1\right).$$

It means that ω_{n,m_3} can be expressed as

$$\omega_{n,m_3} = \left(\frac{\Gamma(n/2 + 1)}{\Gamma((n - m_3)/2 + 1)\Gamma(m_3/2 + 1)} \right)^{1/m_3}.$$

It is well-known that

$$\Gamma(z) = z^{z-1/2} e^{-z} (2\pi)^{1/2} \epsilon_n, \quad \lim_{n \rightarrow \infty} \epsilon_n = 1,$$

so that for any $1 \leq m_3 \leq n$, $n \rightarrow \infty$ we have

$$\begin{aligned} \omega_{n,m_3} &= \\ &= \left(\frac{(n/2 + 1)^{n/2+1/2} e \epsilon_n}{((n - m_3)/2 + 1)^{(n-m_3)/2+1/2} (m_3/2 + 1)^{m_3/2+1/2} (2\pi)^{1/2} \epsilon_{n-m_3} \epsilon_{m_3}} \right)^{1/m_3} \\ &= \left(\frac{(n + 2)^{n/2+1/2}}{(n - m_3 + 2)^{(n-m_3)/2+1/2} (m_3 + 2)^{m_3/2+1/2}} \right)^{1/m_3} \left(\frac{2^{1/2} e \epsilon_n}{2\pi^{1/2} \epsilon_{n-m_3} \epsilon_{m_3}} \right)^{1/m_3} \\ &= \frac{n^{1/2}}{m_3^{1/2+1/(2m_3)}} \\ &\quad \times \frac{(1 + 2/n)^{n/(2m_3)+1/(2m_3)}}{(1 - m_3/n + 2/n)^{n/(2m_3)-1/2+1/(2m_3)} (1 + 2/m_3)^{1/2+1/(2m_3)}} \\ &\quad \times \left(\frac{2^{1/2} e \epsilon_n}{2\pi^{1/2} \epsilon_{n-m_3} \epsilon_{m_3}} \right)^{1/m_3} \asymp \left(\frac{n}{m_3} \right)^{1/2} n^{1/(2m_3)}. \end{aligned} \quad (7)$$

Remark that if $m_3 = \lambda n$, $0 < \lambda < 1$, then $\omega_{n,m_3} \sim (\lambda e)^{-1/2}$ as $n \rightarrow \infty$.

From (6) it follows that for any $L_{m_2} \subset \mathbb{R}^n$ there is such $\alpha^* \in L_{m_2}$ that

$$\|\alpha^*\|_{(2)} \geq \left(\frac{2C_2}{C_1 M_V} \right)^{n/m_3} \omega_{n,m_3} \|\alpha^*\|_V. \quad (8)$$

Recall that $m_3 = \dim(L_{m_1} \cap L_{m_2})$. Since $\alpha^* \in L_{m_1}$ then from (2) we get

$$\|\alpha^*\|_{(2)} \leq C_3 M_{W^\circ} \left(\frac{n}{n - m_1} \right) \|\alpha^*\|_W. \quad (9)$$

Finally, comparing (8) and (9) we find

$$\|\alpha^*\|_V \leq \left(\frac{C_1 M_V}{2C_2} \right)^{n/m_3} \frac{C_3 M_{W^\circ}}{\omega_{n,m_3}} \left(\frac{n}{n - m_1} \right) \|\alpha^*\|_W. \quad (10)$$

In particular, let $m_1 = \mu_1 n$ and $m_2 = \mu_2 n$ for some fixed $\mu_1 > 0$ and $\mu_2 > 0$, $1 < \mu_1 + \mu_2 < 2$, then from (7) and (10) it follows that

$$\|\alpha^*\|_V \leq C(M_V)^{1/(\mu_1 + \mu_2 - 1)} M_{W^\circ} \|\alpha^*\|_W,$$

where $C > 0$ is an absolute constant.

4 Flat Polynomials on \mathbb{M}^d

Let Ω be a compact space with a normalized measure ν , Fix an orthonormal system $\Xi = \{\xi_k\}_{k \in \mathbb{N}} \subset L_2(\Omega, \nu)$ and a sequence $\{k_j\}_{j \in \mathbb{N}}$ such that $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$. Let

$$\Xi^j := \text{lin}\{\xi_k\}_{k=k_j}^{k_{j+1}-1}, \quad \Omega_m := \{j_1, \dots, j_m\}, \quad \Xi(\Omega_m) := \text{lin}\{\Xi^{j_s}\}_{s=1}^m.$$

Put $n := \dim \Xi(\Omega_m) = \sum_{s=1}^m k_{j_{s+1}} - k_{j_s} = \sum_{s=1}^m d_{j_s}$, where $d_{j_s} := \dim \Xi^{j_s}$. Consider the coordinate isomorphism

$$J : \mathbb{R}^n \rightarrow \Xi(\Omega_m)$$

that assigns to $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ the function $J\alpha = \xi^\alpha = \sum_{l=1}^n \alpha_l \xi_{j_l} \in \Xi(\Omega_m)$. Let X be a Banach space such that $\Xi(\Omega_m) \subset X$ for any $\Omega_m \subset \mathbb{N}$. Put $X_n = \Xi(\Omega_m) \cap X$. The definition $\|\alpha\|_{(X_n)} = \|\xi^\alpha\|_X$ induces a norm on \mathbb{R}^n . Put

$$B_{(X_n)}^n := \{\alpha \in \mathbb{R}^n, \|\alpha\|_{(X_n)} \leq 1\},$$

then $B_{X_n}^n := JB_{(X_n)}^n$.

A Banach lattice X is q -concave, $q < \infty$ if there is a constant $C_q > 0$ such that

$$\left(\sum_{i=1}^n \|x_i(\cdot)\|_X^q \right)^{1/q} \leq C_q(X) \left\| \left(\sum_{i=1}^n |x_i(\cdot)|^q \right)^{1/q} \right\|_X$$

for any $n \in \mathbb{N}$ and any sequence $\{x_i(\cdot)\}_{i=1}^n \subset X$ (see, e.g., [20], p. 46).

We will need the following statement.

Lemma 1. *For any $\xi \in \Xi(\Omega_m)$, $m \in \mathbb{N}$ we have*

$$\|\xi\|_{L_p(\Omega, \nu)} \leq C n^{(1/p-1/q)+} \|\xi\|_{L_q(\Omega, \nu)},$$

where $1 \leq p, q \leq \infty$ and $n := \dim \Xi(\Omega_m)$.

Proof Consider the function

$$K_n(x, y) := \sum_{\xi_k \in \Xi(\Omega_m)} \xi_k(x) \overline{\xi_k(y)}.$$

Clearly,

$$K_n(x, y) = \int_{\Omega} K_n(x, z) K_n(z, y) d\nu(z)$$

and $K_n(x, y) = \overline{K_n(y, x)}$. Hence,

$$\|K_n(\cdot, \cdot)\|_{L_\infty(\Omega, \nu)} \leq \|K_n(y, \cdot)\|_{L_2(\Omega, \nu)} \|K_n(x, \cdot)\|_{L_2(\Omega, \nu)}$$

for any $x, y \in \Omega$ and $\|K_n(x, \cdot)\|_{L_2(\Omega, \nu)} \leq C n^{1/2}$, since $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$. It means that

$$\|K_n(\cdot, \cdot)\|_{L_\infty(\Omega, \nu)} \leq C n. \quad (11)$$

Let $\xi \in \Xi(\Omega_m)$, then applying Hölder inequality and (11) we get

$$\|\xi\|_{L_\infty(\Omega, \nu)} \leq \|K_n(\cdot, \cdot)\|_{L_\infty(\Omega, \nu)} \|\xi\|_{L_1(\Omega, \nu)},$$

or

$$\|I\|_{L_1(\Omega, \nu) \cap \Xi(\Omega_m) \rightarrow L_\infty(\Omega, \nu) \cap \Xi(\Omega_m)} \leq Cn,$$

where I is an embedding operator. Trivially, $\|I\|_{L_p(\Omega, \nu) \cap \Xi(\Omega_m) \rightarrow L_p(\Omega, \nu) \cap \Xi(\Omega_m)} = 1$, $1 \leq p \leq \infty$. Hence, using Riesz-Thorin interpolation Theorem and an embedding arguments for any $\xi \in \Xi(\Omega_m)$ we obtain

$$\|\xi\|_{L_p(\Omega, \nu)} \leq Cn^{(1/p-1/q)+} \|\xi\|_{L_q(\Omega, \nu)}, \quad 1 \leq p, q \leq \infty,$$

where

$$(a)_+ := \begin{cases} a, & a \geq 0, \\ 0, & a < 0. \end{cases}$$

In the case $\Omega_m = \{1, \dots, m\}$ the estimates of respective Lévy means have been obtained in [19]. Using Lemma 1 we can generalize our result to an arbitrary index set Ω_m .

Theorem 2. *Let $(\Omega, \nu, \Xi, \{k_l\}) \in \mathcal{K}$ and X is a 2-concave, then for an arbitrary Ω_m ,*

$$M(\mathbb{R}^n, \|\cdot\|_{(X_n)}) \leq C_X, \quad X_n = X \cap \Xi(\Omega_m) \quad (12)$$

where $n := \dim \Xi(\Omega_m)$ and $C_X > 0$ is independent on $n \in \mathbb{N}$. In particular,

$$M(\mathbb{R}^n, \|\cdot\|_{(L_p(\Omega, \nu) \cap \Xi(\Omega_m))}) \leq C \begin{cases} p^{1/2}, & p < \infty, \\ (\log n)^{1/2}, & p = \infty, \end{cases} \quad (13)$$

where $C > 0$ is an absolute constant.

Remark that different estimates of Lévy means have been obtained in [15] - [18]. We are prepared now to prove main result of this article.

Theorem 3. *Assume that $\max\{M_{j^{-1} \circ (X \cap \Xi_n)}, M_{j^{-1} \circ (Y \cap \Xi_n)}\} < C$ for any $n \in \mathbb{N}$ and some absolute constant $C > 0$ and $B_X \subset B_{L_2(\mathbb{M}^d)}$. Then in any subspace $J^{-1} \circ L_s \subset \Xi(\Omega_m)$ there exists such polynomial t_n^* that*

$$\|t_n^*\|_X \leq C_{X,Y} \|t_n^*\|_Y. \quad (14)$$

In particular, the inequality (14) is valid if $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$, X is 2-concave and $\|\cdot\|_{(B_Y \cap \Xi_n)^\circ} \leq \|\cdot\|_{J^{-1} \circ (Y_1 \cap \Xi_n)}$ for some 2-concave Y_1 and any $n \in \mathbb{N}$. Let $(\Omega, \nu, \Xi, \{k_j\}_{j \in \mathbb{N}}) \in \mathcal{K}$, $X = L_p(\Omega, \nu)$, $Y = L_q(\Omega, \nu)$, $1 \leq p, q \leq \infty$, then for an arbitrary spectrum Ω_m , $n = \dim \Xi(\Omega_m)$ and any subspace $J^{-1} \circ L_s \subset \Xi(\Omega_m)$ there exists a polynomial $t_n^ \in J^{-1} \circ L_s$ such that*

$$\|t_n^*\|_{L_p(\Omega, \nu)} \leq C \varrho_n \|t_n^*\|_{L_q(\Omega, \nu)},$$

where

$$\varrho_n = \begin{cases} 1, & 1 < q, p < \infty, \\ (\log n)^{1/2}, & 1 \leq q \leq p < \infty, \\ (\log n)^{1/2}, & 1 < q \leq p \leq \infty, \\ \log n, & 1 \leq q \leq p \leq \infty \end{cases}$$

and $C > 0$ is an absolute constant.

Proof Applying Theorem 2 and Theorem 1 for a fixed $\epsilon \in (0, 1)$ and the inequality

$$\|\cdot\|_{(B_Y \cap \Xi_n)^\circ} \leq \|\cdot\|_{Y_1 \cap \Xi_n},$$

where Y_1 is a 2-concave, we get

$$\frac{\|\alpha^*\|_V}{\|\alpha^*\|_W} \leq CM_V^{1+\epsilon} M_{W^\circ} \leq C_{X, Y_1}$$

Hence, using (12), for any $L_s \subset \mathbb{R}^n$, $s = \dim L_s \geq \epsilon n$, $\epsilon \in (0, 1)$, one can find such $\alpha^* \in L_s$ that

$$\|\alpha^*\|_V \leq C_{X, Y_1} \|\alpha^*\|_W.$$

It means that in any subspace $J \circ L_s$ there exists such $t_n^* \in J \circ L_s \subset \Xi(\Omega_m)$ that

$$\|t_n^*\|_X \leq C_{X, Y_1} \|t_n^*\|_Y.$$

In the case $X = L_p(\Omega, \nu)$ and $Y = L_q(\Omega, \nu)$ we use (13) to get a similar estimate

$$\|t_n^*\|_{L_p(\Omega, \nu)} \leq C_{\varrho_n} \|t_n^*\|_{L_q(\Omega, \nu)},$$

where $C > 0$ is an absolute constant.

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